

HANF NUMBER FOR THE STRICTLY STABLE CASES

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ABSTRACT. Suppose $\mathbf{t} = (T, T_1, p)$ is a triple of two theories in vocabularies $\tau \subseteq \tau_1$ of cardinality λ and a τ_1 -type p over the empty set; in the main case it is stable. We show the Hanf number for the property: “there is a model M_1 of T_1 which omits p , but $M_1 \upharpoonright \tau$ is saturated” is larger than the Hanf number of $L_{\lambda^+, \kappa}$ but smaller than the Hanf number of $\mathbb{L}_{(2^\lambda)^+, \kappa}$ when T is stable with $\kappa = \kappa(T)$. In fact, we characterize the Hanf number of \mathbf{t} when we fix (T, λ) where T is a first order complete, $\lambda \geq |T|$ and demand $|T_1| \leq \lambda$.

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§ 0. INTRODUCTION

§ 0(A). Background on Results.

This continues papers of Baldwin-Shelah, starting from a problem of Newelski [New12] concerning the Hanf number described above for classes $\mathbf{N}_{\lambda,T}$ (defined formally in 1.1). They showed in [BlSh:958] that with no stability restriction the Hanf number is essentially equal to the Löwenheim number of second order logic and in [BlSh:992] showed that for superstable T it is bigger than the Hanf number of $\mathbb{L}_{(2^\lambda)^+, \aleph_0}$ but it is smaller than $\mathbb{L}_{\beth_2(\lambda)^+, \aleph_0}$.

Our original aim was to deal with the case where T is a stable theory and concentrate on the strictly stable case (i.e. stable not superstable).

However, we ask a stronger question.

Question 0.1. Fix a complete first order theory T can cardinal λ , what is $\sup\{H(\mathbf{t}) : \mathbf{t} \text{ as above with } T_{\mathbf{t}} = T, \text{ i.e. belongs to } \mathbf{N}_{\lambda,T} \text{ from 1.1(1)}\}$.

Clearly this is a considerably more ambitious question. We give a quite complete answer. For T strictly stable, our original case, it appears that only the cardinals $|T|, \kappa(T)$ and a derived Boolean Algebra $\mathbb{B}(T)$ of cardinality $|D(T)|$, and the truth, see 0.7 are important. In fact, for any T , we have to add just the truth value of $(2^{\aleph_0} > |T|) \wedge |D(T)| > |T| \cap (T \text{ superstable})$.

Here the infinitary logic $\mathbb{L}_{\lambda^+, \kappa}$ is central.

A major point here is to deal abstractly with what is essentially the Boolean algebra of formulas over the empty set. We introduce in Definition 1.5 the logics $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ where $\mathbb{B} = \mathbb{B}_T$, the members of the Boolean algebra (i.e. formulas from $\mathbb{L}(\tau_T)$) are coded by elements of the model and the union of these logics over the relevant \mathbb{B} 's is called $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$, moreover $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}} = \mathbb{L}_{\lambda, \kappa}[\mathbb{B}_\lambda^{\text{fr}}]$, see below. Then in Observation 1.6(1) we note that

$$H(\mathbb{L}_{\lambda^+, \kappa}) \leq H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) \leq H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}) \leq H(\mathbb{L}_{(2^\lambda)^+, \kappa}).$$

The main result shows that there is an exact equivalence between classes of the form $\mathbf{N}_{\lambda,T}$ and classes of the form $\text{Mod}_\psi, \psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ for \mathbb{B} the Boolean Algebra formulas over the emptyset in T .

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§ 0(B). Preliminaries.

Here for a first order complete T we define the relevant parameters; $\kappa(T), \mathbb{B}_T$ and quote characterization of saturation.

Notation 0.2. 1) τ is a vocabulary.

1A) T denotes a first order theory in $\mathbb{L}_{\tau(T)}$, $\tau_T = \tau(T)$ the vocabulary of T .

2) $\bar{x}_{[u]} = \langle x_i : i \in u \rangle$, similarly $\bar{y}_{[u]}$; e.g. $\bar{x}_{[\alpha]} = \langle x_i : i < \alpha \rangle$.

3) $\mathbb{L}_{\lambda, \kappa}$ for $\lambda \geq \kappa$ is the logic where $\mathbb{L}_{\lambda, \kappa}(\tau) =$ the language = set of formulas, is the closure of the set of atomic formulas under negation, conjunction of the form $\bigwedge_{\alpha < \gamma} \varphi_\alpha, \gamma < \lambda$ and quantification $(\exists \bar{x}_{[u]})\varphi$ where $u \in [\kappa]^{<\kappa}$ but every formula has

$< \kappa$ free variables.

4) Let \mathbb{B} denote a Boolean Algebra and $\text{uf}(\mathbb{B})$ the set of ultra-filters of \mathbb{B} .

- 5) Let \mathbf{t} denote an object as in Definition 1.1 below.
 6) For a theory T let Mod_T be the class of models of T .

Recall

Definition 0.3. Let T be a first order complete theory.

- 1) $\kappa(T)$ is the minimal κ such that: if $A \subseteq M_* \in \text{Mod}_T$ and $p \in \mathbf{S}(A, M)$ then there is $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over B , see [Sh:f, Ch.III].
- 2) Let $\kappa_r(T) = \min\{\kappa : \kappa \text{ regular} \geq \kappa(T)\}$ so $\kappa_r(T)$ is the minimal regular κ such that T is stable in λ whenever $\lambda = \lambda^{<\kappa} + 2^{|T|}$, see [Sh:c, Ch.III].
- 3) Let $\lambda(T)$ be the minimal λ such that T is stable in λ , that is $[M \models T, \|M\| \leq |T| + \aleph_0 \Rightarrow |\mathbf{S}(M)| \leq \lambda]$, see [Sh:c, Ch.III, §5, §6].
- 4) $D_m(T) = \{\text{tp}(\bar{a}, 0, M) : \bar{a} \in {}^m M \text{ and } M \models T\}$ and $D(T) = \bigcup_m S \upharpoonright D_m(T)$.
- 5) Let $\text{EQ}_T = \{\varphi(\bar{x}_{[n]}, \bar{y}_{[n]}) : n < \omega, \varphi \in \mathbb{L}(\tau_T) \text{ and for every model } M \text{ of } T, \{(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in M \text{ and } M \models \varphi[\bar{a}, \bar{b}]\} \text{ is an equivalence relation on } {}^n M \text{ with finitely many equivalent classes}\}$.
- 6) M is \aleph_ε -saturated when for every triple (b, A, N) satisfying $A \subseteq M \prec N, b \in N, A$ finite, some $b' \in M$ realizes the type $\{\varphi(x, b; \bar{a}) : \bar{a} \subseteq A, \varphi(x, y, \bar{a}) \text{ is an equivalence relation with finitely many equivalence classes}\}$, equivalently $\text{stp}(b, A)$, see [Sh:c, Ch.III].

Remark 0.4. 1) Note that by [Sh:c, Ch.III, §5, §6], see 0.11 we have $\min\{\lambda : T \text{ stable in } \lambda\} = \lambda(T) = (\lambda'(T))^{<\kappa(T)} = |\text{uf}(\mathbb{B}_{T^{\text{eq}}})|^{<\kappa(T)}$, see 0.7(2),(3),(4), there is a point in cardinal arithmetic except when $|D(T)| \leq |T| < 2^{\aleph_0}, T$ superstable, T unstable in $|T|$, hence $\lambda'(T) = |D(T)| < 2^{\aleph_0} = \lambda(T)$.

The point is that by [Sh:c, Ch.III].

Fact 0.5. $\{\lambda : T \text{ has a saturated model of cardinality } \lambda\} = \{\lambda : \lambda \geq \lambda(T) \text{ and } \lambda = \lambda^{<\kappa(T)}\}$.

Note that

Observation 0.6. For every Boolean Algebra \mathbb{B}_1 of cardinality $\leq \lambda$ and $\kappa \leq \lambda^+$ there is a Boolean Algebra \mathbb{B}_2 of cardinality λ such that $|\text{uf}(\mathbb{B}_2)| = \Sigma\{|\text{uf}(\mathbb{B}_1)|^\theta : \theta < \kappa\}$, see justification below.

Proof. If $|\mathbb{B}_1| = \lambda, \kappa = \theta^+, \theta \leq \lambda$ we define the Boolean Algebra \mathbb{B} as the free product of θ copies of \mathbb{B}_1 .

If κ is a limit cardinal $\leq \lambda, |\mathbb{B}_1| = \lambda$ let $\mathbb{B}_{2,\theta}$ be as above for $\theta < \kappa$ and \mathbb{B}_2 the disjoint sum of $\langle \mathbb{B}_{2,\theta} : \theta < \kappa \rangle$ so essentially except one ultrafilter, all ultrafilters on \mathbb{B}_2 are ultrafilters on some $\mathbb{B}_{2,\theta}$ so $\text{uf}(\mathbb{B}_2) = 1 + \sum_{\theta < \kappa} \text{uf}(\mathbb{B}_{2,\theta})$. $\square_{0.6}$

Definition 0.7. 1) For a model M and formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{y})} M$ let $\varphi(M, \bar{a}) = \{\bar{b} \in {}^{\ell g(\bar{x})} M : M \models \varphi[\bar{b}, \bar{a}]\}$.

2) For a model $M, \mathbb{B}_{M,m}$ is the Boolean Algebra of subsets of ${}^m M$ consisting of the sets $\{\varphi(M) : \varphi = \varphi(\bar{x}_{[m]})\}$.

2A) $\mathbb{B}_{T,m}$ for $T = \text{Th}(M)$ is the Boolean Algebra of formulas $\varphi(\bar{x}_{[m]})$ modulo equivalence over T , so $\varphi_1(\bar{x}_{[m]}) \leq \varphi_2(\bar{x}_{[m]})$ iff $T \vdash \varphi_1(\bar{x}_{[m]}) \rightarrow \varphi_2(\bar{x}_{[m]})$, elements are actually $\varphi(\bar{x}_{[m]}) / \equiv_T$.

3) Let $\mathbb{B}_M = \langle \mathbb{B}_{M,m} : m < \omega \rangle$; abusing notation let $\text{uf}(\mathbb{B}_M) = \bigcup_m \text{uf}(\mathbb{B}_{M,m})$. Similarly with T instead of M , also below.

3A) Let \mathbb{B}_M be the direct sum of $\langle \mathbb{B}_{M,m} : m < \omega \rangle$ so $\langle 1_{\mathbb{B}_{M,m}} : m < \omega \rangle$ be a maximal antichain of \mathbb{B}_M , $\mathbb{B}_M \restriction \{x \in \mathbb{B}_M : x \leq 1_{\mathbb{B}_{M,m}}\} = \mathbb{B}_{M,m}$ and $\cup\{\mathbb{B}_{M,m} : m < \omega\}$ generates \mathbb{B}_M . Let $\text{tr} - \text{fil}(\mathbb{B}_M)$ = the ultrafilter of \mathbb{B}_M disjoint to $\{1_{\mathbb{B}_{M,n}} : n < \omega\}$ and let $\text{uf}^-(\mathbb{B}_M) = \text{uf}(\mathbb{B}_M) \setminus \{\text{tr} - \text{fil}(\mathbb{B}_M)\}$, ($\text{tr} - \text{fil}$ stands for trivial filter).

4) Let $\lambda'(M)$ be the cardinality of $\text{uf}(\mathbb{B}_M)$.

5) Let $\mathbb{B}_\lambda^{\text{fr}}$ be the Boolean algebra generated freely by $\{\mathbf{a}_\alpha : \alpha < \lambda\}$ so $\text{uf}(\mathbb{B}_\lambda^{\text{fr}})$ has cardinality 2^λ .

Remark 0.8. We may be interested in the Boolean Algebra of formulas which are almost over \emptyset , i.e. $\varphi(\bar{x}, \bar{a}), \bar{a} \in {}^{\ell g(\bar{y})}M$ where $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ satisfies: $\varphi(\bar{x}, \bar{y})$, for some $\vartheta(\bar{x}_m, \bar{y}_m) \in \text{EQ}_M^n$, we have $M \models (\forall \bar{z})(\forall \bar{x}_m, \bar{y}_m)[\vartheta(\bar{x}_m, \bar{y}_m) \rightarrow (\varphi(\bar{x}_m, \bar{z}) \equiv \varphi_n(\bar{y}_m, \bar{z})]$.

But this is not necessary here.

Observation 0.9. 1) $\mathbb{B}_{M,m}$ essentially depend just on $\text{Th}(M)$, i.e. if $T = \text{Th}(M)$ then $\mathbb{B}_{M,m}$ is isomorphic to $\mathbb{B}_{T,m}$ by $\varphi(\bar{x}_{[m]}) + \mathbb{L}(\tau_T) \Rightarrow \mathbf{j}(\varphi(M)) = \varphi(\bar{x}_{[m]}) / \equiv_T$, so $\lambda'(T)$ is well defined.

2) Similarly for other notions from Definition 0.7.

3) $\text{uf}(\mathbb{B}_M), \text{uf}^-(\mathbb{B}_M)$ has the same cardinality, in fact, there is a natural one-to-one mapping π from $\text{uf}(\mathbb{B}_M)$ onto $\text{uf}^-(\mathbb{B}_M)$ such that $D \in \text{uf}(\mathbb{B}_{M,m}) \Rightarrow \pi(D) = \{a \in \mathbb{B}_{M,m} : a \cap 1_{\mathbb{B}_{M,m}} \in D\}$.

Recall ([Sh:c, Ch.III])

Fact 0.10. Let T be a stable (first order complete) theory, $\kappa = \kappa(T)$ and M is an uncountable model of T . Then M is saturated iff

Case 1: $\kappa > \aleph_0$

- (a) if $\mathbf{I} \subseteq M$ is an infinite indiscernible set then there is an indiscernible set $\mathbf{J} \subseteq M$ extending \mathbf{I} of cardinality $\|M\|$
- (b) M is κ -saturated.

Case 2: $\kappa = \aleph_0$

- (a)' if $A \subseteq M$ is finite and $a \in M \setminus \text{acl}(A)$ then there is an indiscernible set \mathbf{J} over A in M based on A such that $a \in \mathbf{J}$ and \mathbf{J} is of cardinality $\|M\|$
- (b)' M is \aleph_ε -saturated (see [Sh:c] or 0.3(6)).

Fact 0.11. Assume T is a stable (first order complete) theory.

- 1) If $\kappa(T) > \aleph_0$ then $\lambda(T) = |D(T)|^{<\kappa_r(T)}$.
- 2) If $\kappa(T) = \aleph_0$ then $\lambda(T)$ is $|D(T)|$ or is $2^{\aleph_0} > |D(T)|$ and for some finite $A \subseteq M$, $M \in \text{Mod}_T$, the set $\{\text{stp}(a, A) : a \in M\}$ has cardinality continuum.

§ 1. THE FRAME

First, we define here $\mathbf{N}_{\lambda,T}$, the set of \mathbf{t} from the abstract when we fix T and for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ we define the class of models $\text{Mod}_{\mathbf{t}}$. We then deal with the logics $\mathbb{L}_{\lambda,\kappa}[\mathbb{B}]$ via which we shall characterize the Hanf number of $\mathbf{N}_{\lambda,T}$ and has the relation amongst such logics.

Definition 1.1. 1) For T complete first order and $\lambda \geq |T|$ let $\mathbf{N}_{\lambda,T}$ be the class of triples $\mathbf{t} = (T, T_1, p) = (T_{\mathbf{t}}, T_{1,\mathbf{t}}, p_{\mathbf{t}})$ such that:

- (a) $T_{\mathbf{t}} = T$
- (b) $T_1 \supseteq T$ is a first order theory and $|\tau(T_1)| \leq \lambda$
- (c) $p(x)$ is an $\mathbb{L}(\tau_{T_1})$ -type.

1A) For \mathbf{t} as above we say $M_1 \models \mathbf{t}$ or $M_1 \in \text{Mod}_{\mathbf{t}}$ or M_1 is a model of \mathbf{t} when:

- (a) $M_1 \models T_{1,\mathbf{t}}$ and M_1 a τ_{T_1} -model
- (b) M_1 omits the type $p_{\mathbf{t}}(x)$
- (c) $M_1 \upharpoonright \tau_T$ is saturated.

1B) Omitting T means: for some T such that $|T| \leq \lambda$.

2) Let $\text{spec}_{\mathbf{t}} = \{\|M\| : M \models \mathbf{t}\}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$.

3) $H(\mathbf{N}_{\lambda,T})$ is the minimal μ such that: if $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ and \mathbf{t} has a model of cardinality $\geq \mu$ then \mathbf{t} has models of arbitrarily large cardinality; see 1.5(3).

4) $\lambda(\mathbf{t}) := \lambda(T_{\mathbf{t}}) + |T_{1,\mathbf{t}}|$.

Convention 1.2. Below $\mathbf{t}, T, T_1, p, \lambda$ are as in Definition 1.1 if not said otherwise and then $\kappa = \kappa_r(T)$ as in 0.3.

Claim 1.3. 1) If $M \in \text{Mod}_{\mathbf{t}}$ has cardinality μ then $\mu = \mu^{<\kappa(T)} + |\lambda(T)|$ hence $\mu \in \text{spec}_{\mathbf{t}} \Rightarrow \mu = \mu^{<\kappa(T)} + \lambda(T)$.

2) If $M \in \text{Mod}_{\mathbf{t}}$ and $\lambda(\mathbf{t}) \leq \mu = \mu^{<\kappa(T)} < \|M\|$ and $A \subseteq M$ is of cardinality μ then for some N we have:

- (a) $N \in \text{Mod}_{\mathbf{t}}$
- (b) $A \subseteq N \prec M$
- (c) N has cardinality μ .

Proof. Easy. Note that also $\mu = \mu^{<\kappa_r(T)}$ hence $\kappa_r(T) \leq \mu$; we choose M_i by induction on $\kappa_r(T)$ such that:

- (a) if i is even then $M_i \prec M$ and $\|M_i\| = \mu$
- (b) if i is odd then $M_i \prec M \upharpoonright \tau(T_{\mathbf{t}})$, $\|M_i\| = \mu$ and M_i is saturated
- (c) if $j < i$ then $|M_j| \subseteq |M_i|$.

There is no problem to carry the induction and then $M' = \cup\{M_{2i} : i < \kappa_r(T)\}$ is as required: $M' \prec M$ by (a)+(c) and Tarski-Vaught, $\|M'\| = \mu$ by cardinal arithmetic and (*) and $M' \upharpoonright \tau(T)$ is saturated by (b)+(c) and [Sh:c, Ch.III]. $\square_{1.3}$

Conclusion 1.4. For understanding the Hanf number of \mathbf{t} , it is enough to consider cardinals $\mu = \mu^{<\kappa(T)} \geq \lambda(\mathbf{t})$.

Now we turn to the logics of the form $\mathbb{L}_{\lambda+,\kappa}[\mathbb{B}]$; first we define them.

Definition 1.5. 1) Assume

- (a) $\lambda \geq \kappa = \text{cf}(\kappa)$
- (b) \mathbb{B} a Boolean Algebra of cardinality λ and let $\text{uf}(\mathbb{B})$ be the set of ultrafilters on \mathbb{B} .

Then

- (α) let $\text{voc}[\mathbb{B}]$ be the class of vocabularies τ of cardinality $\leq \lambda$ such that $c_b \in \tau$ individual constant for $b \in \mathbb{B}$, and $P, Q \in \tau$ unary predicates and $R \in \tau$ binary and τ may have additional signs.
- (β) For $\tau \in \text{voc}[\mathbb{B}]$ let $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$ be the set of sentences $\psi \in \mathbb{L}_{\lambda^+, \kappa}(\tau)$ but we stipulate that from ψ we can reconstruct $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$.

[Note that ψ has $\leq \lambda$ sub-formulas]:

- (γ) omitting τ means $\tau = \tau_\psi$ is the minimal $\tau \in \text{voc}[\mathbb{B}]$ such that $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$, so $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ is essentially a logic.
- 2) For $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$ let $\text{Mod}_\psi^1[\mathbb{B}]$ be the class of models M of ψ (which are τ_ψ -models if not said otherwise) such that (note: clauses (a)-(d) can be expressed in $\mathbb{L}_{\lambda^+, \aleph_0}$, but when $\kappa > \aleph_0$ not clause (e))
 - (a) $P^M = \{c_b^M : b \in \mathbb{B}\}$
 - (b) $\langle c_b^M : b \in \mathbb{B} \rangle$ are pairwise distinct
 - (c) $R \subseteq P^M \times Q^M$
 - (d) for every $a \in Q^M$ the set $\text{uf}^M(a) := \{b \in \mathfrak{B} : M \models c_b R a\}$ belongs to $\text{uf}(\mathfrak{B})$ and if $a_1 \neq a_2 \in Q^M$ then $\text{uf}^M(a_1) \neq \text{uf}^M(a_2)$
 - (e) for every $u \in \text{uf}(\mathbb{B})$ there is one and only one $a \in Q^M$ such that $M \models \bigwedge_{i < \lambda} (c_b R a)^{\text{if}(b \in u)}$, (by (d) unique).
- 3) Let $\text{Mod}_\psi^2[\mathbb{B}]$ be the class of $M \in \text{Mod}_\psi^1[\mathbb{B}]$ such that:
 - (f) $\|M\| = \|M\|^{<\kappa}$ and (follows) $\|M\| \geq |\text{uf}(\mathbb{B})|$.
- 4) For $\iota = 1, 2$ and $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ let $\text{spec}_\psi^\iota[\mathbb{B}] = \{\|M\| : M \in \text{Mod}_\psi^\iota[\mathbb{B}]\}$.
- 4A) Writing $\text{Mod}_\psi^\iota, \text{spec}_\psi^\iota$ we mean $\iota \in \{1, 2\}$ and may omit ι when $\iota = 2$ (because this is the main case for us), see 1.6(0).
- 5) Let $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ be the first μ such that: if $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ and there is $M \in \text{Mod}_\psi[\mathbb{B}]$ of cardinality $\geq \mu$ then $\{\|M\| : M \in \text{Mod}_\psi[\mathbb{B}]\}$ is an unbounded class of cardinals.
- 6) Let $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$ be $\cup\{\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}] : \mathbb{B} \text{ a Boolean}^1 \text{ Algebra of cardinality } \leq \lambda\}$ so every sentence of $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}(T)$ is a sentence in $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$ for some \mathbb{B} as above; so we may stipulate that the set of elements of \mathbb{B} is a cardinal $\leq \lambda$ and $c_i \in \tau$ for $i < \lambda$.
- 7) We define $H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}})$ similarly; yes, this is just $\sup\{H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) : \mathbb{B} \text{ as above}\}$.

Having defined the class of \mathbf{t} 's and the class of models of \mathbf{t} , $\text{Mod}_{\mathbf{t}}$ and their spectrum we should now try to understand the order between them.

¹So every sentence $\psi \in \mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$ fixes a Boolean Algebra \mathbb{B} as above and a vocabulary of cardinality $\leq \lambda$ from $\text{voc}[\mathbb{B}]$ as described.

Observation 1.6. Let \mathbb{B} be a Boolean Algebra of cardinal $\leq \lambda$ and $\kappa \leq \lambda^+$.

0) In the Definition 1.5(5) of $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ it does not matter if we use $\text{Mod}_\psi^1[\mathbb{B}]$ or $\text{Mod}_\psi^2[\mathbb{B}]$.

1) We have $H(\mathbb{L}_{\lambda^+, \kappa}) \leq H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) \leq \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}^{\text{fr}}] = H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}) < H(\mathbb{L}_{(2^\lambda)^+, \kappa})$.

1A) If $\mathbb{B}_\lambda^{\text{pr}}$ is the free Boolean Algebra of cardinality λ from 0.7(5) and $\kappa = \aleph_0$ then $H(\mathbb{L}_{\lambda^+, \kappa}) < \beth_{(2^\lambda)^+} < H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_\lambda^{\text{fr}}])$.

1B) $H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}) < H(\mathbb{L}_{(2^\lambda)^+, \kappa})$.

2) For every $\mu < H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ we have $2^\mu < H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ hence $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ is a strong limit cardinal.

3) If $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ and $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) \leq \sup\{\|M\| : M \in \text{Mod}_\psi[\mathbb{B}]\}$ then $\infty = \sup\{\|M\| : M \in \text{Mod}_\psi[\mathbb{B}]\}$.

4) Like part (2) for $H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}})$.

5) Like part (3) for $\psi \in \mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$ and $\text{Mod}_\psi^{\text{ba}}$.

6) For every $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ there are $\psi_2, \psi_2', \psi_2'' \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ such that:

(a) $\text{spec}_{\psi_2}^1[\mathbb{B}] = \{\mu : \mu = \mu^{<\kappa} \in \text{spec}_{\psi_1}^1[\mathbb{B}]\}$ and²

(b) $\text{spec}_{\psi_2'}^1[\mathbb{B}] = \{\mu^{<\kappa} : \mu \in \text{spec}_{\psi_1}^1[\mathbb{B}]\}$ and

(c) $\text{spec}_{\psi_2''}^1[\mathbb{B}] = \{\mu : \mu \geq \lambda \text{ and } \mu \in \text{spec}_{\psi_1}^1[\mathbb{B}]\}$.

Proof. 0) First, as the Hanf number is $> 2^\lambda \geq |\text{uf}(\mathbb{B})|$, we can ignore models of cardinality $< 2^\lambda$ if $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ then by clause (b) of part (6) there is $\psi_2' \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ we have $\sup(\text{spec}_{\psi_1}^1) < \infty = \sup(\text{spec}_{\psi_1}^1) \leq \sup(\text{spec}_{\psi_2'}^2) < \infty$, hence the Hanf number using spec_ψ^1 's is \leq the Hanf number using spec_ψ^2 's. Moreover, above we get $\sup(\text{spec}_{\psi_1}^1) \leq \sup(\text{spec}_{\psi_2'}^2) = \sup(\text{spec}_{\psi_2'}^1)$ as $\text{spec}_{\psi_2'}^2 = \text{spec}_{\text{spec}_{\psi_2'}^1}^1$ so also the other inequality holds.

1) The first inequality " $H(\mathbb{L}_{\lambda^+, \kappa}) \leq H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ ", see the definitions of $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$. The second inequality, " $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) \leq H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_\lambda^{\text{fr}}])$ " use 1.9(2)(a). The third inequality, " $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_\lambda^{\text{fr}}]) = \mathbb{L}(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}})$ " use the definition of the latter and the second inequality. The fourth inequality, " $H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}) < H(\mathbb{L}_{(2^\lambda)^+, \aleph_0})$ ", see part (1B) holds as every model M satisfying $\lambda \geq \|M\| + |\tau_M|$ can be characterized up to isomorphism by some $\psi \in \mathbb{L}_{(2^\lambda)^+, \kappa}$.

1A) The first inequality " $H(\mathbb{L}_{\lambda^+, \kappa}) < \beth_{(2^\lambda)^+}$ " holds, e.g. by [Sh:c, Ch.VII, §5] recalling $\kappa = \aleph_0$. The second inequality " $\beth_{(2^\lambda)^+} < H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_\lambda^{\text{fr}}])$ " because (as above and) 1.9(2). The third inequality, " $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]) \leq H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_\lambda^{\text{fr}}])$ " holds by 1.9(2)(b), the fourth appears in part (1).

1B) Let $\mathbf{K}_{\lambda^+, \kappa}$ be the class of pairs (ψ, \mathbb{B}) such that \mathbb{B} is a Boolean Algebra of cardinality $\leq \lambda, \psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$. For $(\psi, \mathbb{B}) \in \mathbf{K}_{\lambda^+, \kappa}$ let $H(\psi, \mathbb{B}) = \cup\{\mu^+ : \mu \in \text{spec}_\psi^2(\mathbb{B})\}$. Clearly up to isomorphism (of vocabularies) $\mathbf{K}_{\lambda^+, \kappa}$ has cardinality $\leq 2^\lambda$ hence $\mathbf{C}_{\lambda^+, \kappa} := \{H(\psi, \mathbb{B}) : (\psi, \mathbb{B}) \in \mathbf{K}_{\lambda^+, \kappa}\}$ as cardinality $\leq 2^\lambda$. So let $\langle (\psi_i, \mathbb{B}_i) : i < 2^\lambda \rangle$ be such that (ψ_i, \mathbb{B}_i) as above and $\mathbf{C}_{\lambda^+, \kappa} \setminus \{\infty\} = \{\mu_i : i < 2^\lambda\}$ where $\mu_i = H(\psi_i, \mathbb{B}_i) = \cup\{\mu^+ : \mu \in \text{spec}_{\psi_i}^2[\mathbb{B}_i]\}$. Now we can find $\psi \in \mathbb{L}_{(2^\lambda)^+, \kappa}$ such that $M \models \psi$ iff

(*) $<^M$ is a linear order of $|M|$ and for arbitrarily large $a \in M$ there is $N \in \text{Mod}_{\psi_i}^2[\mathbb{B}_i]$ with universe $\{b : b <^M a\}$.

²Recall that if $\mu > 2^{<\kappa}$ then $(\mu^{<\kappa})^{<\kappa} = \mu$, see [Sh:g].

Together with part (2) below we are done.

2) For any $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ we can find $\psi_2 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ such that $\tau_{\psi_1} \subseteq \tau_{\psi_2}$, $P_*, R_* \in \tau_{\psi_2} \setminus \tau_{\psi_1}$ are unary, binary predicates respectively and:

- (a) if $M_2 \in \text{Mod}_{\psi_2}^{\iota}[\mathbb{B}]$ then
 - $(M_2 \upharpoonright P_*^{M_2} \upharpoonright \tau_{\psi_1}) \in \text{Mod}_{\psi_1}[\mathbb{B}]$
 - $|P_*^{M_2}| \leq \|M_2\| \leq 2^{|P_*(M_2)|}$
- (b) for every $M_1 \in \text{Mod}_{\psi_1}^{\iota}[\mathbb{B}]$ and $\mu = \mu^{<\kappa} \in [\|M_1\|, 2^{\|M_1\|}]$ there is $M_2 \in \text{Mod}_{\psi_2}^1[\mathbb{B}]$ of cardinality μ .

This clearly suffices.

3)-6) Left to the reader. $\square_{1.6}$

The following 1.7, 1.9 is another way to represent the logic $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$ equivalently of the logic $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_{\lambda}^{\text{fr}}]$, hence eventually to state the Hanf numbers.

Definition 1.7. 1) Let $\mathbb{L}_{\lambda^+, \kappa}^*$ be defined like $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$, see 1.1(3) replacing $\langle c_b : b \in \mathbb{B} \rangle$ by $\langle c_i : i < \lambda \rangle$.

2) For $\psi \in \mathbb{L}_{\lambda^+, \kappa}^*$ let Mod_{ψ}^* be defined as in 1.5(1A),(2),(3) replacing $\text{uf}(\mathbb{B})$ by $\mathcal{P}(\lambda)$.

3) Let $H(\mathbb{L}_{\lambda^+, \kappa}^*)$ be defined like $H(\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}])$ in 1.5(5).

4) For $\psi \in \mathbb{L}_{\lambda^+, \kappa}^*$ let $\text{spec}_{\psi}^* = \{\|M\| : M \in \text{Mod}_{\psi}^*\}$; for transparency we will stipulate that from ψ we can reconstruct $\mathbb{L}_{\lambda^+, \kappa}^*$.

Remark 1.8. The following claim essentially tells us that for determining the Hanf number of $\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}$, we may use the “worst” Boolean Algebra, $\mathbb{B}_{\lambda}^{\text{fr}}$ and $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_{\lambda}^{\text{fr}}]$ is essentially equal to $\mathbb{L}_{\lambda^+, \kappa}^*$.

Claim 1.9. 1) The parallel of 1.6 holds for $H(\mathbb{L}_{\lambda^+, \kappa}^*)$.

2) Recall $\mathbb{B}_{\lambda}^{\text{fr}}$ is the Boolean Algebra generated freely by λ generators

- (a) for every Boolean algebra \mathbb{B}_1 of cardinality λ or just $\leq \lambda$ and $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_1]$ there is $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_{\lambda}^{\text{fr}}]$ such that $\text{spec}_{\psi_1}^{\iota} \setminus 2^{\lambda} = \text{spec}_{\psi}^{\iota} \setminus 2^{\lambda}$ for $\iota = 1, 2$
- (b) $H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}[\mathbb{B}_1]) \leq H(\mathbb{L}_{\lambda^+, \kappa}^{\text{ba}}[\mathbb{B}_{\lambda}^{\text{fr}}])$ for \mathbb{B}_1 as above
- (c) for every $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}^*$ there is $\psi_2 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_{\lambda}^{\text{fr}}]$ such that $\{\|M\| : M \in \text{Mod}_{\psi_1}^{\text{ba}}\} = \{\|M\| : M \in \text{Mod}_{\psi_2}^*[\mathbb{B}]\}$, that is $\text{spec}_{\psi_1}^* = \text{spec}_{\psi_2}[\mathbb{B}]$
- (d) for every $\psi_2 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_{\lambda}^{\text{fr}}]$ there is $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}^*$ which are as in clause (c).

Proof. Should be clear. $\square_{1.9}$

Next we have to connect those logics with first order T 's. The easy part is to start with a Boolean Algebra \mathbb{B} and construct a related T .

Claim 1.10. 1) For every Boolean Algebra \mathbb{B} of cardinality $\leq \lambda$ and cardinal $\kappa \leq \lambda^+$ there is $T = T_{\mathbb{B}, \kappa}^1$ such that:

- (*)₁ (a) T is a first order complete and stable
- (b) $|T| = \lambda$ and $\kappa(T) = \kappa$
- (c) $\lambda(T)$ is the cardinality of $\text{uf}[\mathbb{B}_T]$, in fact, \mathbb{B}_T is not much more complicated than \mathbb{B} but we shall not elaborate, see 1.15 below

- (d) T has elimination of imaginaries.

Moreover

- (*)₂ (a) $\tau = \tau_{\mathbb{B}, \kappa} = \{P_b : b \in \mathbb{B}\} \cup \{Q_\theta : \theta < \kappa \text{ is infinite}\} \cup \{E_{\theta, i} : \theta \leq \kappa \text{ is infinite, } i < \theta\}$ where P_b, Q_θ are unary predicates, $E_{\theta, i}$ a binary predicate
- (b) T is the model completion of the universal theory T_0 , where a τ -model M satisfied T_0 iff
- (α) $b \mapsto P_b^M$ embeds \mathbb{B} into the Boolean Algebra $\mathcal{P}(P_{1_{\mathbb{B}}}^M)$ so $P_{0_{\mathbb{B}}}^M = \emptyset$
- (β) $\langle P_{1_{\mathbb{B}}}^M \rangle^\wedge \langle Q_\theta^M : \theta < \kappa \rangle$ are pairwise disjoint
- (γ) $E_{\theta, i}^M$ is an equivalence relation on Q_θ^M so $a E_{\theta, i}^M b \Rightarrow a \in Q_\theta^M$
- (ε) if $i < j < \theta$ then $E_{\theta, j}^M$ refines $E_{\theta, i}^M$.

2) For $\mathbb{B}, \lambda, \kappa$ as above there is $T = T_{\mathbb{B}, \kappa}^2$ such that:

- (*)₁ (a), (b) as above
- (c) $\lambda(T) = \lambda + 2^{\aleph_0}$
- (*)₂ as above but
- (a) we add $Q_0, E_{0, n} (n < \omega)$ with Q_0 unary $E_{0, n}$ binary
- (b) (β) also Q_0^M is disjoint to them
- (ζ) $E_{0, n}^M$ is an equivalence relation on P_0^M
- (η) $E_{0, 0}^M$ has one equivalence class
- (θ) $E_{0, n+1}^M$ refines $E_{0, n}^M$ and divides each $E_{0, n}^M$ equivalence classes to at most 2.

Proof. Easy. □_{1.9}

We like to translate “ $M \models \psi, \psi \in \mathbb{L}_{\lambda^+, \kappa}$ ” to $M \in \text{Mod}_{\mathbf{t}}$, that is, when $\kappa > \aleph_0$ and $\kappa(T) \geq \kappa$. However, the following is the “translation of $\psi \in \mathbb{L}_{\lambda^+, \kappa}(\tau_0)$ ”; i.e. it deals strictly with the logic $\mathbb{L}_{\lambda^+, \kappa}$; in particular a Boolean Algebra \mathbb{B} is not present. Our aim is to do some of work of 1.13 in which we are really interested. So 1.11 is not directly related to \mathbf{t} ’s! as there is no saturation requirement; moreover stability does not appear in 1.11, 1.13.

Note that in 1.11 we can let κ_1 be such that $\kappa = \kappa_1^+$ or $\kappa_1 = \kappa$ is a limit cardinal and let $\Upsilon = \kappa_1 + 1$ and omit $F_{\kappa_1}, P_{\kappa_1}$.

Theorem 1.11. *The $\mathbb{L}_{\lambda^+, \kappa}$ -representation Theorem*

Assume $\psi \in \mathbb{L}_{\lambda^+, \kappa}(\tau_0)$, so of course, $|\tau_0| \leq \lambda$. Let Υ be κ if $\kappa \leq \lambda$ and $\kappa + 1$ if $\kappa = \lambda^+$.

Then we can find a tuple $(\tau_1, T_1, p(x), \bar{F}, \bar{P})$ such that (for \bar{F}, \bar{P} as below):

- (A) (a) τ_1 is a vocabulary $\supseteq \tau_0$ of cardinality λ
- (b) \bar{F} a sequence of unary function symbols with no repetitions of length Υ , new (i.e. from $\tau_1 \setminus \tau_0$), let $\bar{F} = \langle F_i : i < \Upsilon \rangle$
- (c) \bar{P} a sequence of unary predicates with no repetitions of length Υ , new (i.e. from $\tau_1 \setminus \tau_0$), let $\bar{P} = \langle P_i : i < \Upsilon \rangle$

- (d) T_1 is a first order theory in the vocabulary τ_1
- (e) $p(x)$ is $\{P_*(x) \wedge x \neq c_i : i < \lambda\}$, a $\mathbb{L}(\tau_1)$ -type (even quantifier-free), so P_* is a unary predicate and c_i for $i < \lambda$ individual constants
- (B) the following conditions on a τ_0 -model M_0 are equivalent
 - (a) $M_0 \models \psi$ and $\|M_0\| = \|M_0\|^{<\kappa} + \lambda$
 - (b) there is a τ_1 -expansion M_1 of M_0 to a model of T_1 omitting $p(x)$ such that:
 - (α) $\langle P_i^{M_1} : i < \Upsilon \rangle$ is a partition of $|M_1|$
 - (β) if $i < \Upsilon$ and $a_j \in M_1$ for $j < i$ then for some $b \in P_i^{M_1}$ we have $j < i \Rightarrow F_j^{M_1}(b) = a_j$.

Proof. Note that as ψ has no free variables, without loss of generality every subformula of ψ has a set of free variables equal to $\{x_i : i < \varepsilon\}$ for some $\varepsilon < \kappa$.

Let Δ be the set of subformulas of ψ so without loss of generality (a syntactical rewriting) there is a list $\langle \varphi_i(\bar{x}_{[\varepsilon(i)]}) : i < i(*) \leq \lambda \rangle$ of Δ such that $\varepsilon(0) = 0$, $\varphi_0 = \psi$ and $\bar{x}_{[\varepsilon(i)]}$ is a sequence of length $< \kappa$ of variables, in fact, without loss of generality $\bar{x}_{[\varepsilon(i)]} = \langle x_\varepsilon : \varepsilon < \varepsilon(i) \rangle$ and $\varepsilon(i) < \kappa$.

For any τ_0 -model M such that $\|M\| = \|M\|^{<\kappa} + \lambda$ we say N codes M when:

- (*) (a) N expands M
 - (b) $\langle F_i^N : i < \Upsilon \rangle, \langle P_i^N : i < \kappa \rangle$ satisfies (B)(b)(α), (β) of the claim (with N instead of M_1)
 - (c) $Q_i^N = \{b \in P_{\varepsilon(i)}^N : M \models \varphi_i(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i) \rangle)\}$ for $i < i(*)$
 - (d) $\langle c_i^N : i < \lambda \rangle$ are pairwise distinct and $P_*^N = \{c_i^N : i < \lambda\}$
 - (e) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = \bigwedge_{j < j(i)} \varphi_{i,j}(\bar{x}_{\varepsilon(i)})$ so $\varphi_{i,j}(\bar{x}_{\varepsilon(i)}) = \varphi_{\mathbf{i}(i,j)}(\bar{x}_{\varepsilon(\mathbf{i}(i,j))})$ and so $\varepsilon(\mathbf{i}(i,j)) = \varepsilon(i)$ then $F_{1,i} \in \tau(N)$ is unary and for $b \in P_{\varepsilon(i)}^N$ we have:
 - (α) $N \models "F_{1,i}(b) = c_j \wedge \neg \varphi_i(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i) \rangle)"$ implies $M \models \neg \varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i) \rangle)$ which means:
if $\varphi_{i,j} = \varphi_{\mathbf{i}(i,j)}$ and $N \models "\neg Q_i(b) \wedge c_j = F_{1,i}(b)"$ then $M \models "\neg Q_{\mathbf{i}(i,j)}[b]"$
- and, of course
- (β) if $M \models \varphi_i(\langle f_\varepsilon(b) : \varepsilon < \varepsilon(i) \rangle)$ and $j < \varepsilon(i)$ then $M \models \varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i) \rangle)$
 - (f) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = (\exists \bar{x}_{[\varepsilon(i), \zeta(i)]}) \varphi_{j_1(i)}(\bar{x}_{\varepsilon(i)}, \bar{x}_{[\varepsilon(i), \zeta(i)]})$ and $F_\varepsilon(b) = a_\varepsilon$ for $\varepsilon < \varepsilon(i)$ then (α) \Leftrightarrow (β) where
 - (α) $M_1 \models \varphi_1[\langle a_\varepsilon : \varepsilon < \varepsilon(i) \rangle]$
 - (β) $M_1 \models (\exists y) \varphi_{j_1(i)}(\langle a_\varepsilon : \varepsilon < \varepsilon(i) \rangle, \langle F_\zeta(y) : \zeta \in [\varepsilon(i), \zeta(i)] \rangle)$.

Now let

- ⊞ (a) τ_1 is $\tau_\psi \cup \{F_\varepsilon, P_\varepsilon : \varepsilon < \Upsilon\} \cup \{Q_i : i < i(*)\} \cup \{F_{1,i} : i < i(*)$ and φ_i is a conjunction
- (b) $T_1 = \cap \{\text{Th}(N) : \text{there is } M \text{ a } \tau_0\text{-model of } \psi \text{ such that } \|M\| = \|M\|^{<\kappa} + \lambda \text{ and } N \text{ code } M\}$ or write explicitly all that is used
- (c) $p(x) = \{P_*(x) \wedge x \neq c_i : i < \lambda\}$.

Now check that

$$\oplus (\tau_1, T_1, p(x), \bar{F}, \bar{P}) \text{ is as required.}$$

□_{1.11}

Remark 1.12. So how does 1.11 help for our main aim? It starts to translate $\psi \in \mathbb{L}_{\lambda^+, \kappa}(\tau_0)$ to $(\tau_1, T_1, p(x))$, so instead having blocks of quantifiers $(\exists \bar{x}_{[\varepsilon]}), \varepsilon < \kappa$ we have $(\exists x)$, i.e. by the function $\langle F_i : i < \varepsilon \rangle$ we code any ε -tuple by one element.

This will help later to make “the $\tau(T_t)$ -reduct is saturated” equivalent to the coding.

Recalling Definition 1.5(6) of $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$, we get the section main result: translating from $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ to a representation naturally more complicated than the one for $\psi \in \mathbb{L}_{\lambda^+, \aleph_0}$.

Theorem 1.13. *The $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ -representation theory*

Assume \mathbb{B} is a Boolean Algebra of cardinality $\leq \lambda$. Then the conclusion of the theorem 1.11 holds using $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}], \text{Mod}_{\psi}^2[\mathbb{B}]$ instead $\psi \in \mathbb{L}_{\lambda^+, \kappa}, \text{Mod}_{\psi}$ but in (B)(b) we add:

- (B) (b) (γ) $c_b(b \in \mathbb{B})$ are individual constants (in $\tau_1 \setminus \tau_0$) with no repetition, $P, Q \in \tau_1$ unary, $R \in \tau_1$ binary
- (δ) $P_*^{M_1} = \{c_b^{M_1} : b \in \mathbb{B}\}$
- (ε) $R^{M_1} \subseteq P^{M_1} \times Q^{M_1}$
- (ζ) for every $b \in Q^{M_1}$ the set $u(b, M_1) := \{b \in \mathbb{B} : (c_b, b) \in R^{M_1}\}$ is an ultrafilter of \mathbb{B}
- (η) for every ultrafilter D of the Boolean Algebra \mathbb{B} there is one and only one $b \in Q^{M_1}$ such that $u(b, M_1) = D$.

Proof. Similar to 1.11.

□_{1.13}

Remark 1.14. 1) The only non-“ $\mathbb{L}_{\lambda^+, \kappa}$ demand” in (B) is in (b)(η), the existence, this is not expressible by a sentence of $\mathbb{L}_{\lambda^+, \kappa}$, even with extra predicates.

2) As indicated above, $\mathbb{B}_{\lambda}^{\text{fr}}$ is the “worst, most complicated Boolean Algebra” for our purpose. So it is natural to wonder about the order among the relevant Boolean Algebras, so 1.15, 1.16 try to deal with it.

Definition 1.15. 1) We define a two-place relation $\leq_{\lambda^+, \kappa}^*$ among the Boolean Algebras \mathbb{B} of cardinality $\leq \lambda$

$\mathbb{B}_1 \leq_{\lambda, \kappa}^* \mathbb{B}_2$ iff: there is a sentence $\psi_2 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_2]$ unary, predicates $P_1, Q_1 \in \tau_{\psi}$ and binary predicate R_2 and individual constants $c_b^1(b \in \mathbb{B}_1)$ from τ_{ψ} such that:

- if $M \models \psi_2$ then $P_1^M = \{c_b^M : b \in \mathbb{B}_1\}$ and $R_1^M \subseteq P_1^M * Q_1^M$ satisfies the demands in 1.5(2).

2) We let $\equiv_{\lambda, \kappa}^*$ be defined by $\mathbb{B}_1 \equiv_{\lambda^+, \kappa}^* \mathbb{B}_2$ iff $\mathbb{B}_1 \leq_{\lambda^+, \kappa}^* \mathbb{B}_2$ and $\mathbb{B}_2 \leq_{\lambda^+, \kappa}^* \mathbb{B}_1$.

Claim 1.16. 1) $\leq_{\lambda^+, \kappa}^*$ is a quasi-order on the class of Boolean Algebras of cardinality $\leq \lambda$.

2) Hence $\equiv_{\lambda, \kappa}^*$ is an equivalence relation.

3) In 1.10(1) we have $\mathbb{B}_T \equiv_{\lambda^+, \kappa}^* \mathbb{B}$ where $T = T_{\lambda, \kappa}^1$.

4) If $\mathbb{B}_1 \leq_{\lambda, \kappa}^* \mathbb{B}_2$ then for every $\psi_1 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_1]$ there is $\psi_2 \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_2]$ such that:

- (a) $\text{spec}_{\psi_1}^2 = \text{spec}_{\psi_2}^2$
- (b) *if M_1 is a $\tau(\psi_1)$ -model then $M_1 \in \text{Mod}_{\psi_1}^2$ iff $M_1 = M_2 \restriction \tau_{\psi_1}$ for some $M_2 \in \text{Mod}_{\psi_2}^2$; pedantically we should have an embedding π of τ_{ψ_1} into τ_{ψ_2} and demand $M_1 = (M_2 \restriction \text{Rang}(\pi))^{\pi}$, naturally defined.*

Proof. Should be clear.

□1.16

§ 2. REAL EQUALITY FOR EACH T

§ 2(A). Answering the Original Question and the New One.

The original question was about the strictly stable case, i.e. fixing $\kappa > \aleph_0$, dealing with $\{\mathbf{t} \in \mathbf{N}_\lambda : \kappa(T_{\mathbf{t}}) = \kappa\}$.

Theorem 2.1. *Assume T is a stable first order complete of cardinality $\leq \lambda$ and $\kappa = \kappa_r(T) = \min\{\theta : \theta \text{ regular and } \theta \geq \kappa(T)\}$ and $\lambda(T) = \min\{\lambda : T \text{ stable in } \lambda\}$ and let $\mathbb{B} = \mathbb{B}_T$, see Definition 0.7(3A).*

Assume further that $\kappa(T) > \aleph_0$ (i.e. T is not superstable).

- 1) Then $\{\text{spec}_{\mathbf{t}} : \mathbf{t} \in \mathbf{N}_{\lambda,T}\} = \{\text{spec}_{\psi}^2[\mathbb{B}] : \psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]\}$.
- 2) If $\tau_0 = \tau_T$ and $\psi_0 = \wedge\{\varphi : \varphi \in T\}$ or just $\tau_T \subseteq \tau_0, |\tau_0| \leq \lambda, \psi_0 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_0)$ and $M \in \text{Mod}_{\psi_0}[\mathbb{B}] \Rightarrow M \models T$ then there is $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ such that $\text{spec}_{\psi_0}^2[\mathbb{B}] = \text{spec}_{\mathbf{t}}$.
- 3) If $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ then for some $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_1), \tau_1 \supseteq \tau(T_2)$ and $\text{spec}_{\psi_1}^2[\mathbb{B}] = \text{spec}_{\mathbf{t}}$.

Remark 2.2. The proof gives more: that the two contexts have the same PC classes. This section is divided to two subsections each to one direction.

Proof. 1) By parts (2),(3).

2) By §(2B) below.

3) By §(2A) below, i.e. by 2.10 noting 2.9. □_{2.1}

Conclusion 2.3. *If T is first order complete stable theory, $\kappa = \kappa(T), |T| \leq \lambda$ then $H(\mathbf{N}_{\lambda,T})$ is bigger than $H(\mathbb{L}_{\lambda^+,\kappa})$ but smaller than $H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.*

Proof. First assume T is strictly stable, i.e. $\kappa(T) > \aleph_0$. The “bigger than $H(\mathbb{L}_{\lambda^+,\kappa})$ ” follows by 2.1(2) recalling 1.6(1), the first inequality. The “smaller than $H(\mathbb{L}_{(2^\lambda)^+,\kappa})$ ” follows by 2.1(3) recalling 1.6(1), the second and third inequality. We are left with the case T is superstable, but then we quote [BSh:992], or see below. □_{2.3}

Now we turn to the general case, first we divide to cases in 2.4, prove that fixing λ and $|T|$ all the cases occur in 2.5. Then in each case we give quite a complete answer.

Claim 2.4. *If T is a complete first order theory of cardinality $\leq \lambda$ then the pair (λ, T) satisfies exactly one of the following cases:*

Case A: T is unstable

Case B1: T is strictly stable (i.e. stable not superstable)

Case B2: T superstable and $\lambda(T) \leq \lambda$

Case B3: T superstable $\lambda(T) > \lambda$ but $2^{\aleph_0} \leq \lambda$ hence $(D(T)) = |\text{uf}(\mathbb{B}_T)| = \lambda(T)$

Case B4: T superstable, $\lambda(T) > \lambda, 2^{\aleph_0} > \lambda$ but $|D(T)| > |T|$ hence $\mathbb{B}_{\aleph_0}^{\text{fr}}$ is embeddable into \mathbb{B}_T

Case C: T superstable, $\lambda(T) > \lambda, 2^{\aleph_0} > \lambda, |D(T)| \leq |T|$

Proof. Should be clear except the “hence” in Case B3 which holds by [Sh:c, Ch.III,§5], see the proof of 2.5 below (the hence in case B4 is easier). □_{2.4}

Claim 2.5. 1) For any cardinal λ the pairs $\{(\lambda, T) : T \text{ complete first order of cardinal } \leq \lambda\}$ all the cases from 2.4 occurs, modulo the restriction on λ .
 2) Moreover, if \mathbb{B} is a Boolean Algebra of cardinal $\leq \lambda$ then all the cases of 2.4 occur among the pairs $\{(\lambda, T) : T \text{ first order complete of cardinality } \leq \lambda \text{ and } T \text{ stable} \Rightarrow \mathbb{B}_T \equiv_{\lambda^+, \kappa(T)}^* \mathbb{B}\}$ again modulo the restriction on λ and $|\text{uf}(\mathbb{B})|$ (see the proof).

Proof. For case A use T the theory of random graphs.

For the other cases fix \mathbb{B} .

For case B1, see claim 1.10, for any $\kappa \in [\aleph_1, \lambda^+]$.

For case B2, we should assume $|\text{uf}(\mathbb{B})| \leq \lambda$ and we use claim 1.10 for $\kappa = \aleph_0$.

For case B3, by [Sh:c, Ch.III, §5] we know that $\lambda(T) = |D(T)|$.

For case B4, similarly.

For case C, let $T_0 = \text{Th}(\omega 2, E_n)_{n < \omega}$, $E_n = \{(\eta, \nu) : \eta, \nu \in \omega 2 \text{ and } \eta \upharpoonright n = \nu \upharpoonright n\}$.

□_{2.5}

Theorem 2.6. Assume (λ, T) satisfies one of the cases B1-B4 from 2.4.

- 1) $\{\text{spec}_{\mathbf{t}}^2 : \mathbf{t} \in \mathbf{N}_{\lambda, T}\}$ is equal to $\{\text{spec}_{\psi}^2 : \psi \in \mathbb{L}_{\lambda^+, \kappa_r(T)}[\mathbb{B}_T]\}$.
- 2) $\{H(\mathbf{t}) : \mathbf{t} \in \mathbf{N}_{\lambda, T}\} = \{H(\psi) : \psi \in \mathbb{L}_{\lambda^+, \kappa_r(T)}[\mathbb{B}_T]\}$, see Definition xxx,yyy.
- 3) $H(\mathbf{N}_{\lambda, T}) = H(\mathbb{L}_{\lambda^+, \kappa_r(T)}[\mathbb{B}_T])$ hence is $\geq H(\mathbb{L}_{\lambda^+, \kappa_r(T)})$ and is $< H(\mathbb{L}_{(2^\lambda)^+, \kappa_r(T)})$ by xxx.
- 4) If (λ, T') satisfies one of the cases B1-B4 and for transparency $\kappa_r(T') = \kappa_r(T)$ and $\mathbb{B}_{T'} \leq_{\lambda^+, \kappa_r(T)}^* \mathbb{B}_T$ then $\{\text{spec}_{\mathbf{t}}^2 : \mathbf{t} \in N_{\lambda, T'}\} \subseteq \{\text{spec}_{\mathbf{t}}^2 : \mathbf{t} \in N_{\lambda, T}\}$.

Proof. Similar to the proof of Theorem 2.1, except that: for case B1 use Claim 2.10, for case B2 use Claim 2.11, for Case B3 use claim 2.12(1) and for Case B4 use 2.12(2). □_{2.6}

Theorem 2.7. If (λ, T) satisfies Case C of 2.4, then the parallel of Theorem 1.13 holds except that we replace \mathbb{B}_T by $\mathbb{B}'_T = \mathbb{B}_T \oplus \mathbb{B}_{\aleph_0}^{\text{fr}}$, oplus is the sum.

Proof. Similar, we elaborate in §(2D). □_{2.7}

Theorem 2.8. Assume (λ, T) satisfies Case A of Claim 2.4 and $\mathbf{C} = \{\mu : \mu = \mu^{<\mu} \geq |D(T)|\}$ is an unbounded class of cardinals. Recall that

(*) if M is a saturated model of T then $\|M\| \in \mathbf{C}$, see [Sh:c, Ch.III].

Then the results of [BlSh:958] holds even if we fix T , that is: $\{\text{spec}_{\mathbf{t}} : \mathbf{t} \in \mathbf{N}_{\lambda, T}\} = \{\text{spec}_{\psi} \cap \mathbf{C} : \psi \in \mathbb{L}_{\lambda^+, \lambda^+}^{\text{mse}}\}$ where $\mathbb{L}_{\lambda^+, \lambda^+}^{\text{ms.e}}$ is $\mathbb{L}_{\lambda^+, \lambda^+}$ extended by the quantification over 2-place relations of cardinality $<$ smaller than that of the model.

Proof. Of course, the inclusion \subseteq is trivial. For the other inclusion the proof splits to two cases and always at least one of them holds by [Sh:c, Ch.II].

Case 1: T has the independence property

So some $\varphi(x, \bar{y}_n) \in \mathbb{L}(\tau_T)$ has the independence property.

Let $\tau_0 = \tau_T \cup \{P, F_\ell, Q, c_i : i < \lambda, \ell < n, \omega\}$, Q unary, p an n -place predicate, F_ℓ a binary function symbol c_k an individual constant.

Let T_0^* be the $\mathbb{L}(\tau_0)$ -theory such that:

(*) for a τ_0 -model M , $M \models T_0^*$ iff

(a) $M \upharpoonright \tau_T \models T$

- (b) $\{\varphi(x, \bar{a}) : \bar{a} \in P^M\}$ is an independent set of formulas
- (c) the function $(a, b) \Rightarrow \langle F_\ell(a, b) : \ell < n \rangle$ is a one-to-one function from $M \times M$ onto P^M
- (d) $c_i^M \in Q^M$ are pairwise disjoint?
- (e) $<^M$ is a linear order of (M)
- (f) $<^M$ satisfies the schemes of being well ordered (in fact enough for some specific schemes).

Note

- (*) if M is a saturated model of T then it has an expansion M'_0 to a τ_0^* -model of T_0 such that $Q^M = \{c_i^M : i < \lambda\}$.

The rest should be clear being as in [BSh:958]; that is, (informally) for suitable \mathbf{t} we are

- guaranteed that in $M \in \text{Mod}_{\mathbf{t}}$ the saturation tells us we can represent enough sets such that together with type omitted we know $<^M$ is a well ordering so can define a subset of order type $\|M\|$ and then can express ψ .

Case 2: T has the strict order property

There are n and $\varphi(\bar{x}_{[n]}, \bar{y}_{[n]}) \in \mathbb{L}(\tau_T)$ which defines a partial order with arbitrarily large finite chains. Let $M \models T$ be \aleph_1 -saturated so we can find $\bar{a}_k \in {}^n M$ for $k \in \mathbb{N}$ such that $M \models \varphi[\bar{a}_{k(1)}, \bar{a}_{k(2)}]$ iff $k(1) < k(2)$. We expand M to M_0 by an interpretation of number theory c :

- $P_0^M = \{\bar{a}_k : k \in \mathbb{N}\}$
- $F_T^{M_0}$ is the partial function $F_T^{M_0}(\bar{a}_{k(1)}, \bar{a}_{k(2)}) = \bar{a}_{k(1)+k(2)}$
- $F_x^{M_0}$ is the partial function $F_x^{M_0}(\bar{a}_{k(1)}, \bar{a}_{k(2)}) = \bar{a}_{k(1) \cdot k(2)}$
- $<^{M_0} = \{(\bar{a}_{k(1)}, \bar{a}_{k(2)}) : k(1) < k(2)\}$.

Let M'_0 be a model of $\text{Th}(M_0)$ such that $|P_0^{M'_0}| = |M'_0|$, let M''_0 be M'_0 expanded by unary function $F_\ell^{M''_0}(\ell < n)$ such that $a \mapsto \langle F_\ell^{M''_0}(a) : \ell < n \rangle$ is a one-to-one function from M'_0 onto $P^{M''_0}$.

Lastly, let $T' = \text{Th}(M''_0)$.

Now by Malliaris-Shelah [?]

- (*) if N is a model of T' such that $\aleph_N := (P_0^N, <^N, F_+^N, F_x^N)$ then
 - (a) \aleph_N is a model of PA
 - (b) if in addition $N \upharpoonright \tau_T$ is saturated then $(P_0^N, <^N)$ is an atomically saturated model.

Hence by Malliaris-Shelah [MiSh:998]

- (*) above also \aleph is saturated.

Using $\langle F_\ell : \ell < n \rangle$ we can continue as in Case 1.

□_{2.8}

§ 2(B). **Given** $\mathbf{t} \in \mathbf{N}_{\lambda,1}$.

Hypothesis 2.9. For this subsection we are given $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ such that T is complete first order stable so $\lambda \geq |T|$ and let $\mathbb{B} = \mathbb{B}_T, \kappa = \kappa_\tau(T)$, without loss of generality:

- (a) $P, Q, R, c_b (b \in \mathbb{B})$ are not in $\tau(T_1)$ and with no repetition
- (b) P, Q are unary predicates, R is a binary predicate, c_b individual constants
- (c) $\tau_2 = \tau(T_1) \cup \{P, Q, R, c_b : b \in \mathbb{B}\}$.

Claim 2.10. Assume $\kappa > \aleph_0$. There is $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau_1)$ such that $\text{Mod}_{\mathbf{t}} = \{N \mid \tau(T_1) : N \models \psi \text{ so } \tau(N) = \tau(\psi) \supseteq \tau_1\}$.

Proof. Note this below proving 2.11, 2.12 we use this proof stating the changes; there $\kappa(T) = \aleph_0$, i.e. T is superstable.

Stage A:

Without loss of generality we can replace T by T^{eq} (no need for new elements: we can extend T_1 to have a copy of M^{eq} with new predicates and an isomorphism). The use of T^{eq} is anyhow just for transparency. For $\theta = \text{cf}(\theta) < \kappa_r(T)$ choose a sequence $\bar{\varphi}_\theta = \langle \varphi_{\theta,i}(x, \bar{y}_{\theta,i}) : i < \theta \rangle$ witnessing $\theta < \kappa_r(T)$ equivalently $\theta < \kappa(T)$.

Stage B:

Let $\tau = \tau(T_1) \cup \{P, Q, R, S_{\varphi(\bar{x}_n, \bar{y}_n)}, G_n, c_b, Q_\theta, <_\theta, F_i, P_i, F_{1,i} : b \in \mathbb{B}, i < \kappa, \varphi(\bar{x}_n, \bar{y}_n) \in \text{EQ}_T\}$, see Definition 0.3(4) on EQ_T ; where the union is without repetitions, G binary function, P, Q unary predicates, c_b individual constant, R binary predicate, $S_{\varphi(\bar{x}_n)}$ an n -place function for $\varphi(\bar{x}_n) \in \mathbb{L}(\tau_T), F_i$ unary function for $i < \kappa; F_{1,n}$ is an n -place function symbol, G_n an n -place function symbol.

For awhile fix $M_1 \in \text{Mod}_{\mathbf{t}}$, note that by 0.5

$$(*)_1 \quad \|M_1\| = \|M_1\|^{<\kappa} \geq \lambda(T).$$

Let $M = M_1 \upharpoonright \tau(T)$ and let $\mathcal{M}[M_1]$ be the set N of such that

- (*)₂ (a) N is a τ -expansion of M_1
- (b) $P^N, Q^N, R, \langle c_b^N : b \in \mathbb{B} \rangle$ code \mathbb{B}_T and $\text{uf}(\mathbb{B}_T)$, see 0.7(3) and e.g. 1.13(B)(b)(γ) – (η) and
- (c)(α) $S_{\varphi(\bar{x}, m)}^N(\bar{a}) = \{c_b^N\}$ when $M \models \varphi_b[\bar{a}]$ essentially this says $b = \varphi_b(x_{[m]}) / \equiv_T$ for $b \in \mathbb{B}_{T,m}$
- (β) $Q^N = \{d_D : D \in \text{uf}(\mathbb{B}_T)\}$ and $R^N = \{(c_b^N, d_D) : b \in \mathbb{B} \text{ and } D \in \text{uf}(\mathbb{B}), b \in D\}$
- (d) for every $i < \kappa$ and $\bar{a} = \langle a_j : j < i \rangle \in {}^i M$ for some $b \in N$ we have $(\forall j < i)(F_j^N(b) = a_j)$ and $b \in P_i^N$
- (e) $\langle P_i^N : i < \lambda \rangle$ is a partition of N
- (f)(α) $F_{1,m}^N$ is a function from ${}^m M = {}^m M$ to Q^N such that if $\bar{a} \in {}^m M$ then $d = F_{1,m}^N(\bar{a})$ is the member of Q^N coding $\text{tp}(\bar{a}, \emptyset, M)$, i.e.
 - if $\varphi(\bar{x}_{[m]}) \in \mathbb{L}(\tau_T)$ and $b = \varphi(\bar{x}_{[m]}) / \equiv_T$ then we have $(\bar{a}, c_b^N) \in S^N$ iff $M \models \varphi[\bar{a}]$
- (β) if $D \in \text{uf}(\mathbb{B}_{T,m})$ then for some $\bar{a} \in {}^m M, F_{1,m}^N(\bar{a}) = d_D$ recall $\mathbb{B}_{T,m} \subseteq \mathbb{B}_T$

- (g) for any regular $\theta < \kappa_r(T)$ we have:
- (α) $Q_\theta^N = \cup \{P_i^N : i \leq \theta\}$ and $(Q_\theta^N, <_\theta^N)$ is a partial order which is a tree with θ levels isomorphic to $({}^\theta \geq \|M_1\|, \triangleleft)$ say $\pi_\theta : {}^\theta \geq \|M_1\| \rightarrow Q_\theta^{\text{tr}}$ is such an isomorphism
 - (β) let $\bar{a}_\eta^\theta = \langle F_i^N(\pi_\theta(\eta)) : i < \ell g(\bar{y}_{\theta,i}) \rangle$ for $\eta \in {}^\theta \geq \|M_1\|$
 - (γ) $b_1 <_\theta^N b_2$ iff for some $i_1 < i_2 < \theta$ we have $b_1 \in P_{i_1}^N, b_2 \in P_{i_2}^N$ and $j < \ell_1 \Rightarrow F_j^N(b_1) = F_j^N(b_2)$
 - (δ) if $i < \theta, \eta \in {}^i \|M_1\|$ and $\alpha < \beta < \|M_1\|$ then $N \models \neg(\exists x)((\varphi_{\theta,i}(x, \bar{a}_{\eta \hat{\ } \langle \alpha \rangle}^\theta) \wedge \varphi_i(x, \bar{a}_{\eta \hat{\ } \langle \beta \rangle}^\theta))$
 - (ε) if $n < \omega, i_0 < \dots < i_{n-1} < \theta, \eta_k \in {}^{(i_k)} \|M_\ell\|$ and $\eta_0 \triangleleft \eta_1 \triangleleft \dots$ then $N \models (\exists x)(\bigwedge_{k < n} \varphi_{i_k}(x, \bar{a}_{\eta_k}^\theta))$
 - (ζ) $F_{\theta,j,i}(\pi(\eta)) = \pi(\eta \upharpoonright i)$ when $i < j \leq \theta, \eta \in {}^j \|M_1\|$
 - (θ) for every $c \in Q_\theta^N, F_\theta^N(c)$ is $\pi_\theta(\eta)$ for some $\eta \in {}^\theta \geq \|M_1\|$ letting $j_\eta = \ell g(\eta)$ we have
 - if $i < j_\eta$ then $N \models \varphi_{\theta,i}[c, \bar{a}_{\eta \upharpoonright (i_1)}^\theta]$
 - if $j_\eta < \theta$ then $\alpha < \|M_1\| \Rightarrow N \models \neg \varphi_{j_\eta}(c, \bar{a}_{\eta \hat{\ } \langle \alpha \rangle})$
 - (ι) $F_{\theta,2}^N$ is a binary function such that: if $\eta \in {}^\theta \geq \|M_1\|$ then $\langle F_{\theta,i}^N(c, \pi_\theta(\eta)) : c \in \|M_1\| \rangle$ list with no repetitions $\langle \pi_\theta(\eta \hat{\ } \langle \alpha \rangle) : \alpha < \|M_1\| \rangle$
 - (κ) $F_{i,1,\theta}^N$ or $F_{\theta,1}^N$ is a unary function for every $c \in M, F_{1,\theta}(c)$ is
 - $\pi(\eta)$ for some $\eta \in {}^\theta \geq \|M_1\|$ and for any $i \leq \theta, \nu \in {}^i \|M_1\|$ we have c realize $\{\varphi_j(x, \bar{a}_{\nu \upharpoonright j}^\theta) : j < i \text{ iff } \nu \trianglelefteq \eta\}$
 - (h)(α) if $j < \kappa$ has cofinality θ , then we have witnesses for clause (d), i.e. if it holds for every $j_1 < j$ then it holds for j ; that is, choose $\langle i_j(\iota) : \iota < \theta \rangle$, an increasing with limit j and demand: $b_i \in M_2$ for $i < j, d \in N$ and $F_{\theta,2}^N(d) \in P_\theta^N$ and $\iota < \theta \wedge i_* < i_j(\iota) \Rightarrow F_{i_*}^N(F_\iota^N(d)) = b_{i_*}$ then there is $d' \in P_j$ such that $i_* < j \Rightarrow F_*(d') = b_{i_*}$
 - (i)(α) if $\kappa > \aleph_0$ and $\langle a_n : n < \omega \rangle$ is an indiscernible set in M then for³ some $b, a \mapsto G_2^N(a, b)$ is a one-to-one function from M onto an indiscernible set which includes $\{a_n : n < \omega\}$
 - (β) if $\kappa = \aleph_0, \bar{c} \in {}^n M, b \in M$ is not algebraic over \bar{c} , then
 - $a \mapsto G_{n+2}^N(a, b, \bar{c})$ is one-to-one
 - $G_{n+n}^N(b, b, \bar{c}) = \bar{b}$
 - $\{G^N(a, b, \bar{c}) : a \in M\}$ is an indiscernible set over \bar{c} based on \bar{c} , all in M .

Let $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$ be such that:

³note that is $\kappa > \aleph_0$ we can use G a two-place function symbol

- (*)₃ a τ -model N satisfies ψ iff: for a relevant large enough subset Λ of $\mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}](\tau)$ of cardinality $\leq \lambda$, $\psi = \bigwedge \{\varphi \in \Lambda : \text{if } M_1 \in \text{Mod}_{\mathbf{t}} \text{ and } N \in \mathcal{M}[M_1] \text{ then } N \models \varphi\}$, we may alternatively demand (a)-(h) below:
- (a) $N \upharpoonright \tau_T$ is a model of T , moreover
 - (b) $N \upharpoonright \tau_{T_1}$ is a model of T_1
 - (c) $N \upharpoonright \tau_{T_1}$ omits p
 - (d), (e), (f) the parallel of those clauses in (*)₂
 - (g) for every m , every m -type coded by some $a \in \mathbb{B}_{T, m}$ if $b \in P_{2i}$ code $\langle a_j : j < 2i \rangle$ satisfies $\langle a_{2j}, a_{2j+1} : j < i \rangle$ is a τ -elementary mapping and $a_{2i} \in N$ then for some $b' \in P_{2i+1}$ and a_{2i+1} the element b' code the τ -elementary mapping $\langle (a_{2j}, a_{2j+1}) : j \leq i \rangle$
 - (h) recalling $\kappa > \aleph_0$ if $\langle a_n : n < \omega \rangle$ is an indiscernible set then for some $b, a \mapsto G_2^N(a, b)$ is a one-to-one function from N onto an indiscernible set which includes $\{a_n : n < \omega\}$.

Now

- (*)₄ (a) $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ indeed
- (b) every $M_1 \in \text{Mod}_{\mathbf{t}}$ can be expanded to a model for Mod_{ψ}^* (this is more than being a model of ψ !)
- (c) if $N \in \text{Mod}_{\psi}$ then $N \upharpoonright \tau(T_1) \in \text{Mod}_{\mathbf{t}}$.

[Why? For clause (a) read (*)₃. For clause (b) read (*)₂ + (*)₃. For clause (c), first why $M_1 = N \upharpoonright \tau_{T_1}$ is a model of T_1 ? Because $M_1 \in \text{Mod}_{\mathbf{t}} \wedge N \in \mathcal{M}[M_1] \models N \upharpoonright \tau(T_1) = M_1 \models T_2$. Second, why M_1 omit $p_{\mathbf{t}}$? Recalling (*)₂(e)(α) + (β) and choice of ψ this should be clear. Third, why is $M = N \upharpoonright \tau_T$ saturated? It realizes every $p \in D_m(T) = \mathbf{S}^m(\emptyset, M)$, by (*)₃(g), it is κ -sequence-homogeneous by (*)₃(j) hence is κ -saturated. By (*)₃(k), every indiscernible subset \mathbf{I} of cardinal \aleph_0 by (*)₂(g) for $\theta = \aleph_0$ can be extended to one of cardinality $\|M\|$. As $\kappa > \aleph_0$ by the last two sentences, M is saturated by [Sh:c, Ch.III].]

So we are done. □_{2.10}

Claim 2.11. *Like 2.10, but T is superstable and $\lambda(T) \leq \lambda$.*

Proof. Here the proof “why $M = N \upharpoonright \tau_T$ is saturated inside the proof of (*)₄(c) is different. There is a saturated $M_* \in \text{Mod}_T$ of cardinality $\leq \lambda$ and we can demand on ψ that $N \models \psi$ implies M_* is elementarily embeddable into $N \upharpoonright \tau_T$ and $N \upharpoonright \tau_T$ is \aleph_0 -sequence homogeneous.

Note that

- (*) if $M_* \prec M \in \text{Mod}_T$ and M is \aleph_0 -sequence homogeneous implies M is \aleph_{ε} -saturated, see 0.3(0).

In this case (*)₃(b)(β) of the proof of 2.10 implies M is saturated because by [Sh:c, Ch.III]

- (*) M is saturated when: if M is \aleph_{ε} -saturated and for every finite $A \subseteq M$ and $a \in M \setminus \text{acl}(A)$ there is an indiscernible set $\mathcal{I} \subseteq M$ over A of cardinal $\|M\|$ based on A (i.e. $\text{Av}(M, \mathbf{I})$ does not fork over A) to which a belongs.

□_{2.11}

Claim 2.12. 1) Like 2.10 but T is superstable and $2^{\aleph_0} \leq \lambda$.
 2) Like 2.10, but T superstable and $|D(T)| > |T|$.

Proof. As the proof of 2.11 the problem is how ψ guarantees “ $N \upharpoonright \tau_T$ is \aleph_ε -saturated” .
 As the model is \aleph_0 -sequence homogeneous it suffices

- (*) for every m and $D \in \text{uf}(\mathbb{B}_{T,m+1})$ equivalently $p \in D_{m+1}(T)$ for some $\bar{a} \wedge \langle c \rangle \in {}^{m+1}N$ realizing p we have: if $N \upharpoonright \tau_T \prec M'$ and $c' \in M'$ realizes $\text{tp}(c, \bar{a}, N \upharpoonright \tau_T)$ then some $c'' \in N \upharpoonright \tau_T$ realizes $\text{stp}(c', \bar{a}, M')$ in M' .

Let $p = \text{tp}(c, \bar{a}, M)$ and we let $\lambda_* = \lambda(p), \langle E_\alpha(x_0, x_1; \bar{y}_{[m]}) : \alpha < \lambda_* \rangle$, see [Sh:c, Ch.III,5.1,pg.125].

Case 1: $\lambda_* = \aleph_0$

If $2^{\aleph_0} \leq \lambda$ this is easy. If $|D(T)| > |T|$ then for some m there is an independent sequence $\langle \varphi_n(\bar{x}_{[m]}) : n < \omega \rangle$ of formulas of $\mathbb{L}(\tau_T)$ over T ; (that is, if $M \in \text{Mod}_T$ then any non-trivial finite Boolean combination of them is realized in M) and we continue as in the second case.

Case 2: $\lambda_* > \aleph_0$

In this case by [Sh:c, Ch.III,5.10,pg.126,5.11,pg.127] there is a sequence of length λ_* of formulas of the form $\varphi[x, \bar{a}]$ independent in \mathfrak{C}_T . Hence there is an independent over T sequence $\langle \varphi_i(x, \bar{y}_{[m]}) : i < \lambda_* \rangle$ of formulas from $\mathbb{L}(\tau_T)$, so $\mathbb{B}_{\lambda_*}^{\text{fr}}$ is embeddable into $\mathbb{B}_{T,m+1}$. So ψ says that the Boolean Algebra $\mathcal{P}(\lambda_*)$ is interpreted in N for every relevant λ_* , but they are all $\leq |T|$.

From this it is easy to have ψ ensuring (*).

□_{2.12}

§ 2(C). **Coding** $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}_T]$.

Hypothesis 2.13.

- (a) T is a complete first order theory,
- (b) $\lambda \geq |T|, \lambda^+ \geq \kappa$
- (c) $\mathbb{B} = \mathbb{B}_T$.

Claim 2.14. Assume $\psi \in \mathbb{L}_{\lambda^+, \kappa}[\mathbb{B}]$ and $\kappa = \kappa_r(T) < \infty$ so T is stable.

There is $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda, T}$ such that $\tau(T_1) \supseteq \tau(\psi)$ and $\text{Mod}_{\mathbf{t}} = \{N \upharpoonright \tau(\psi) : N \in \text{Mod}_{\psi}[\mathbb{B}]\}$.

Proof. We apply 1.13 to \mathbb{B} and ψ and get $(\tau_1, T_1, p(*), \bar{F}, \bar{P})$ as in 1.11, 1.13 and without loss of generality $\tau_1 \cap \tau(T) = \emptyset$. Now we immitate the proof of 2.10. □_{2.14}

§ 2(D). **Elaborating Case C.**

In §(2B) we treat most theories T but not all. The remaining case is

Hypothesis 2.15.

- ⊞ (a) T is superstable of cardinality λ
- (b) $\lambda(T) > \lambda$
- (c) $2^{\aleph_0} > \lambda$
- (d) $\lambda \geq |D(T)|$.

Claim 2.16. *There are $m, M \in \text{Mod}_T$ and $\bar{a} \in {}^m M$ such that $\{\text{stp}(c, \bar{a}, M) : c \in M\}$ is of cardinality 2^{\aleph_0} .*

Proof. Should be clear. □_{2.16}

Definition 2.17. For any model M and a sequence \bar{a} from M (or a set \subseteq), let $\mathbb{B}_{M, \bar{a}, m}$ is the Boolean Algebra of subsets of ${}^m M$ of the form $\varphi(M, \bar{c})$, where $\varphi(\bar{x}_{[m]}, \bar{z}) \in \mathbb{L}(\tau_M)$, $\bar{b} \in {}^{\ell g(\bar{z})} M$ and $\varphi(\bar{x}, \bar{c})$ is almost over \bar{a} which means: for some $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}) \in \mathbb{L}(\tau_M)$ we have:

- in M , $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a}) \vdash \varphi(\bar{x}_{[m]}, \bar{a}) \equiv \varphi(\bar{y}_{[m]}, \bar{a})$
- $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a})$ defines in M an equivalence relation with finitely many equivalence classes.

Claim 2.18. *For T as in 2.15, letting M, \bar{a}, m be as in 2.16 and $\mathbb{B} = \mathbb{B}_{M, \bar{a}, m}$ the result of 2.10 and Theorem 2.1 hold if we use \mathbb{B} instead of \mathbb{B}_T .*

Proof. As above, really $m = 1$ suffice; in particular if $p \in \mathbf{S}(\bar{a}, M)$, $\bar{a} \in {}^k M$, $M \in \text{Mod}_T$ then $\lambda_*(p) \leq \aleph_0$ (otherwise $|D(T)| \geq 2^{\aleph_0} > |T|$, contradiction). □_{2.18}

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